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# On the Hopf algebraic origin of Wick normal ordering

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**Abstract**

A combinatorial formula of Rota and Stein is taken to perform Wick reordering in quantum field theory. Wick's theorem becomes a Hopf algebraic identity called Cliffordization. The combinatorial method relying on Hopf algebras is highly efficient in computations and yields closed algebraic expressions.

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## 1. Introduction

Quantum field theory needs, due to the quantization of fields, an operator ordering. Using, for example, canonical quantization for bosonic fields $b(\vec{r}, t)$ and fermionic fields $\psi(\vec{r}, t)$ and their canonical conjugates $\Pi_b$ and $\Pi_\psi$ one *assumes* the canonical (anti) commutation relations

$$[\Pi_{b(\vec{r},t)}, b(\vec{r}', t)]_- = \delta(\vec{r} - \vec{r}') \qquad \{\Pi_{\psi(\vec{r},t)}, \psi(\vec{r}', t)\}_+ = \delta(\vec{r} - \vec{r}'). \tag{1}$$

Hence one is forced to introduce ordered monomials into the variables to span the space of the theory. Feynman introduced the time-ordering device in [20]. However, for practical calculations one has to change the ordering to the so-called normal ordering [11, 39]. This transition is seen in terms of Feynman diagrams as passing over to one-particle irreducible graphs: see, for example, [23]. If this is compared with the transition in thermodynamics formulated in the BBGKY (Bogoliubov–Born–Green–Kirkwood–Yvon) hierarchy to Ursell functions [32,37], one can see that the process of transforming a hierarchy from time- to normal-ordered correlation functions removes the two-particle correlations in higher correlation functions. The thermodynamic transition to Ursell functions removes any $(n-1)$-point correlation from $n$-point correlation functions. The well known $\tau_n$ and $\phi_n$ functions [28] for time- and normal-ordered correlation functions are given as

$$\begin{aligned} \tau_n(1, \ldots, n) &:= {}_{\mathrm{Phys}}\langle 0|\mathcal{T}(\psi(\vec{r}_1, t_1), \ldots, \psi(\vec{r}_n, t_n))|0\rangle_{\mathrm{Phys}} \\ \phi_n(1, \ldots, n) &:= {}_{\mathrm{Phys}}\langle 0|\mathcal{N}(\psi(\vec{r}_1, t_1), \ldots, \psi(\vec{r}_n, t_n))|0\rangle_{\mathrm{Phys}} \end{aligned} \tag{2}$$

where ${}_{\mathrm{Phys}}\langle 0| \cdots |0\rangle_{\mathrm{Phys}}$ is the expectation value w.r.t. the actual vacuum of the theory—not a Fock vacuum in general, due to Haag's theorem [22]—and $\mathcal{T}, \mathcal{N}$ are time- and normal-ordering

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operators. The contraction—sometimes called covariance—is denoted as $\mathcal{C}(\psi_{I_1}, \psi_{I_2})$, which could in principle be any scalar-valued function but will be later on assigned as the propagator $F$, see [14, 15, 18, 21]. The hierarchy of $\tau$ and $\phi$ correlation functions is connected via Wick's theorem as

$$\begin{aligned}
\tau_0 \quad &= \langle 0|0\rangle = 1 \\
\tau_1(1) &= \phi_1(1) \\
\tau_2(12) &= \langle 0|\mathcal{T}(\psi_1\psi_2)|0\rangle = \langle 0|\mathcal{N}(\psi_1\psi_2)|0\rangle + \mathcal{C}(\psi_1\psi_2) \\
&= \phi_2(12) + \mathcal{C}(\psi_1\psi_2) \\
\tau_3(123) &= \phi_3(123) + \mathcal{C}(\psi_1\psi_2)\tau_1(3) + \mathcal{C}(\psi_2\psi_3)\tau_1(1) + \mathcal{C}(\psi_3\psi_1)\tau_1(2) \\
&\ \ \vdots
\end{aligned} \tag{3}$$

We have simplified our notation and dropped the subscript Phys from the vacuum. Furthermore, we use *integers as variables* to denote the index sets of the field operators and correlation functions, which may be algebraic or continuous. Now, irreducibility means that an $\tau_n$ function cannot be written as a product of $\tau_{n-r}$ functions ($r \geqslant 1$): for example, $\tau_4(1234) = \tau_2(12)\tau_2(34)$ would be reducible.

If one is interested in composite particle calculations one has to remove lower-order correlations from the desired correlation functions. This can be achieved in a non-perturbative manner as shown in [35]. The Wick reordering theorem can be easily proved by Clifford algebraic methods [12], unveiling its hidden geometric origin.

**Remark.** Caianiello [5] tried some time ago to connect time- and normal-ordered correlation functions using Clifford algebras. Our approach is different, since we do *not* transform the Grassmann wedge products into Clifford products as Caianiello did, but into another *dotted* wedge product, see below. Since the Schwinger sources force the $\tau_n$ and $\phi_n$ functions to be antisymmetric, this cannot be achieved using a Clifford basis without destroying the invariance under basis transformations. Our approach is, however, $\boldsymbol{sl}(n)$ invariant. As long as *normalized* states are considered (that is, quotients of expectation values) this approach is $\boldsymbol{gl}(n)$ invariant.

In this paper we show that the transition from time- to normal ordering is based on a Hopf algebra structure of the Schwinger sources of quantum field theory. We introduce the well known Shuffle–Hopf algebra [38] which might also be called Grassmann–Hopf algebra. This viewpoint is of particular use in our case. Then we use a combinatorial formula of Rota and Stein [34] to show that there are infinitely many Grassmann algebras which are *not* isomorphic as Hopf algebras. It is shown that the *Rota–Stein Cliffordization*—employed in a certain sense—is exactly a closed form of the Wick transformation. This should be compared with the cumbersome recursive process usually performed in such reorderings [23]. The computational benefits of the Hopf algebraic method compared with the usually employed and non-perturbative method of Stumpf is discussed.

## 2. Non-perturbative normal ordering

### *2.1. Using generating functionals*

Driven by needs of composite particle quantum field theory, Stumpf introduced the non-perturbative normal ordering [35]. This method is based on generating functionals—avoiding path integrals for certain reasons—to describe the Schwinger–Dyson hierachy of quantum field theory. A closed formula is then given for the transition from time- to normal ordering and vice versa. However, if actual computations have to be performed, a *replacement* mechanism is effectively used. We define the (fermionic) sources $j_K(\vec{r}, t) \cong j_I \cong j_n$ using abstract indices or even integers to denote the set of relevant quantum numbers—the bosonic case can be handled along the same lines [9, 14, 15, 18]—and their duals $\partial_K(\vec{r}, t) \cong \partial_I \cong \partial_n$ which have to fulfil

$$\{\partial_{I_1}, \partial_{I_2}\} := 0 \qquad \{j_{I_1}, j_{I_2}\} := 0 \qquad \{\partial_{I_1}, j_{I_2}\} := \delta_{I_1 I_2}. \tag{4}$$

Furthermore, we define the functional Fock state for convenience as

$$\partial_I |0\rangle_F = 0 \qquad \forall I. \tag{5}$$

Then we are able to write down the generating functionals which code the hierarchy as

$$|\mathcal{T}(j)\rangle := \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!} \tau_n(1, \ldots, n) j_1 \ldots j_n |0\rangle_F \tag{6}$$

or

$$|\mathcal{N}(j)\rangle := \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!} \phi_n(1, \ldots, n) j_1 \ldots j_n |0\rangle_F. \tag{7}$$

The functional form of quantum field theory may be found in [35] where it is shown that this formalism is able to replace usual methods: for example, the path integral approach.

Let now $F_{K_1 K_2}(\vec{r}_1, t_1, \vec{r}_2, t_2) \cong F_{I_1 I_2}$ be the exact propagator of the theory. One can prove the following theorem [12, 18, 35]:

$$|\mathcal{T}(j)\rangle = \mathrm{e}^{-\frac{1}{2} F_{I_1 I_2} j_{I_1} j_{I_2}} |\mathcal{N}(j)\rangle \qquad |\mathcal{N}(j)\rangle = \mathrm{e}^{\frac{1}{2} F_{I_1 I_2} j_{I_1} j_{I_2}} |\mathcal{T}(j)\rangle. \tag{8}$$

Expanding the series and comparing coefficients of the $j$ sources one arrives at Wick's theorem. The factor $\frac{1}{2}$ was introduced to meet the definitions of [35] and could be absorbed in $F$.

### 2.2. *Using Clifford algebras*

It was shown in a series of publications [14–16, 18] that quantum field theory can be reformulated in terms of infinite-dimensional Clifford algebras of arbitrary bilinear form (see [1, 6, 13, 17, 18, 29, 30]) now called *quantum Clifford algebras* [12]. Essentially, one performs the following steps:

(i) Let $V = \langle j_I \rangle$ be the linear space spanned by the Schwinger sources $j_I$.

(ii) Build the exterior algebra—symmetric algebra for bosons—over this space as the formal polynomial ring in the anticommuting sources

$$\begin{aligned} &\{j_{I_1}, j_{I_2}\}_+ = 0 \qquad && \bigwedge V = \mathbb{C} \oplus V \oplus V \wedge V \oplus \cdots \\ &|\mathcal{T}(j)\rangle \in \bigwedge V && |\mathcal{N}(j)\rangle \in \bigwedge V. \end{aligned} \tag{9}$$

(iii) Define the space $V^*$ of linear forms on $V$ as the span of the dual bases $\partial_I$, i.e. $V^* = \langle \partial_I \rangle$ and build up the dual exterior algebra:

$$V^* = \langle \partial_I \rangle \qquad \{\partial_{I_1}, \partial_{I_2}\}_+ = 0 \qquad \{\partial_{I_1}, j_{I_2}\}_+ = \delta_{I_1 I_2}. \tag{10}$$

This space is defined to be reflexive since the same index set is used for $j$ and $\partial$ bases. We use also the notation $\partial_I = j_I \,\lrcorner_\delta$ where we introduced the contraction. Indeed, this can be written basis-free (for example, $x \,\lrcorner_\delta = x_I \partial_I$) using summation convention for discrete and continuous parts of the index set.

(iv) Extend this setting to an action of $\bigwedge^* V \cong (\bigwedge V)^*$ on $\bigwedge V$ in the following way [6, 12, 13, 18, 25, 29]:

$$
\begin{aligned}
&\text{(i)} \quad \partial_{I_1}(j_{I_2}) = \delta_{I_1 I_2} \\
&\text{(ii)} \quad \partial_I(AB) = (\partial_I A)B + \hat{A}(\partial_I B) \qquad (11)\\
&\text{(iii)} \quad A^*(B^*C) = (A^*B^*)C.
\end{aligned}
$$

The notation is as follows: $A, B, C \in \bigwedge V$, i.e. $A = A_{I_1,\ldots,I_n} j_{I_1} \ldots j_{I_n}$, $A^*, B^* \in \bigwedge V^*$, $\hat{A} = (-1)^r A_{I_1,\ldots,I_r} j_{I_1} \ldots j_{I_r}$ and (note the reversion of indices here) $A \mapsto A^* = A_{I_1,\ldots,I_n} \partial_{I_n} \ldots \partial_{I_1}$. This can be recast entirely—avoiding wedge and contraction—in Clifford algebraic form by index doubling, see [12, 18].

(v) Define the field operators as Clifford algebra elements obtained by the Clifford map according to Chevalley deformation:

$$
\begin{aligned}
\psi_K(\vec{r}, t) \equiv \psi_I &:= \partial_I + B_{I_1 I_2} j_{I_2} \\
&\cong \partial_K(\vec{r}, t) + B_{K_1 K_2}(\vec{r}_1, t_1, \vec{r}_2, t_2) j_{K_2}(\vec{r}_2, t_2) \qquad (12)
\end{aligned}
$$

where summation and integration is once again implicit. This is a particular form of a deformation quantization.

Define now explicitly the wedge product denoted by $\wedge$ as the sign for the product of the Schwinger sources. Furthermore, we use index-free notation to shorten the formulae. Let us now define the 'bi'-vector $F := F_{I_1 I_2} j_{I_1} \wedge j_{I_2}$. We introduce a *second* wedge product, the dotted-wedge [26] denoted by $\dot\wedge$, defining on $x, y \in V$:

$$
\begin{aligned}
x \,\dot\wedge\, y &:= x \wedge y + F \underset{\delta}{\lrcorner} (x \wedge y) \\
&= x \wedge y + F_{xy}. \qquad (13)
\end{aligned}
$$

Observe that $x \,\dot\wedge\, y = -y \,\dot\wedge\, x$ and $\dot\wedge$ is indeed antisymmetric and a proper exterior product which can be extended to the whole algebra $\dot{\bigwedge} V$.

Let $A^\wedge(j, \partial)$ be a polynomial in $j, \partial$ by using the wedge product, i.e. $A^\wedge(j, \partial) = A_0 + A_{I_1} j_{I_1} + A_{K_1} \partial_{K_1} + A_{I_1,K_1} j_{I_1} \wedge \partial_{K_1} + \cdots + A_{I_1\ldots I_n, K_1 \ldots K_m} j_{I_1} \wedge \cdots \wedge j_{I_n} \wedge \partial_{K_1} \wedge \cdots \wedge \partial_{K_m} + \cdots$ and an analogous definition for $A^{\dot\wedge}(j, \partial)$ where the dotted wedge is employed. In [12] the following theorem was proved.

Let $\mathrm{e}^F$ denote the exterior exponential of $F \in \bigwedge^2 V$ and $A^\wedge = A^\wedge(j, \partial)$ an arbitrary operator in $\mathrm{End}(\bigwedge V)$. Then we have

$$
\mathrm{e}^F \wedge A^\wedge(j, \partial) \wedge \mathrm{e}^{-F} = A^{\dot\wedge}(j, \partial) = A^\wedge(j, d) \qquad d := \partial - Fj \qquad (14)
$$

for operators and

$$
\mathrm{e}^F |\mathcal{T}(j)\rangle^\wedge = |\mathcal{N}(j)\rangle^{\dot\wedge} \qquad \mathrm{e}^{-F} |\mathcal{N}(j)\rangle^{\dot\wedge} = |\mathcal{T}(j)\rangle^\wedge \qquad (15)
$$

for functional states. The peculiar feature of this transition is that in a first step *only* the product is transformed. Hence we have

$$
|\mathcal{N}(j)\rangle^{\dot\wedge} = \sum \frac{\mathrm{i}^n}{n!} \tau_n(1, \ldots, n) j_{I_1} \dot\wedge \cdots \dot\wedge j_{I_n} |0\rangle_F. \qquad (16)
$$

The *normal-ordered* functional is thus written with the *time-ordered* correlation functions. The usually obtained normal-ordered correlation functions appear only after we have *rewritten* the normal-ordered functional in terms of the *old wedge* $\wedge$. We arrive at

$$
|\mathcal{N}(j)\rangle^\wedge = \sum \frac{\mathrm{i}^n}{n!} \phi_n(1, \ldots, n) j_{I_1} \wedge \cdots \wedge j_{I_n} |0\rangle_F \qquad (17)
$$

where the $\phi_n$ functions are connected to the $\tau_n$ functions via the Wick theorem w.r.t. the contraction—or covariance—$F$. This transition is called Wick isomorphism $\phi$ and is a $\mathbb{Z}_2$-graded algebra homomorphism [12].

Symbolically we write this as

$$C\ell(V, Q) = \phi \circ C\ell(V, B) \tag{18}$$

where $C\ell(V, Q)$ and $C\ell(V, B)$ are Clifford algebras over the space $V$ w.r.t. the quadratic form $Q$ or the—not necessary symmetric—bilinear form $B$, see [12,13,18]. In the present case, we look at both 'Clifford algebras' as degenerated ones, i.e. as Grassmann algebras: that is, we set $Q \equiv 0$ and let $B = -B^T$ be a totally antisymmetric form, which is also degenerated since $1/2(B + B^T) \equiv 0$.

**Example 1 (Schwinger–Dyson hierarchy for a free Hamiltonian).** Let $H$ be the functional Hamiltonian of a free fermionic field. Such an $H$ takes the form

$$H^{\wedge}(j, \partial) := D_{I_1 I_2} j_{I_1} \partial_{I_2} \tag{19}$$

where $D_{I_1 I_2}$ is the kinetic operator (for example, a d'Alembertian or Laplacian): see [15,18,36] for a detailed model of spinor QED. The normal-ordered operator is obtained as

$$\begin{aligned} H^{\dot\wedge}(j, \partial) &= \mathrm{e}^{F} \wedge H^{\wedge}(j, \partial) \wedge \mathrm{e}^{-F} \\ &= H^{\wedge}(j, d) \\ &= D_{I_1 I_2} j_{I_1} (\partial_{I_2} - F_{I_2 I_3} j_{I_3}) \\ &= D_{I_1 I_2} j_{I_1} \partial_{I_2} - D_{I_1 I_2} F_{I_2 I_3} j_{I_1} j_{I_3} \end{aligned} \tag{20}$$

and the generating functional of the Schwinger–Dyson hierarchy transforms as follows:

$$\begin{aligned} &\mathrm{e}^{F} \wedge H^{\wedge}(j, \partial) |\mathcal{T}(j)\rangle >^{\wedge} = \mathrm{e}^{F} \wedge E |\mathcal{T}(j)\rangle^{\wedge} \\ &H^{\dot\wedge}(j, \partial) |\mathcal{N}(j)\rangle >^{\dot\wedge} = E |\mathcal{N}(j)\rangle^{\dot\wedge} \\ &H^{\wedge}(j, d) |\mathcal{N}(j)\rangle >^{\wedge} = E |\mathcal{N}(j)\rangle^{\wedge}. \end{aligned} \tag{21}$$

This example shows that finally the transition from time- to normal ordering is given by re-expressing the dotted wedge $\dot\wedge$ in terms of the undotted wedge $\wedge$ and vice versa. This can be achieved by the above formal substitution in the operators (for example, $H^{\dot\wedge}(j, \partial) = H^{\wedge}(j, d)$) using (14) and expanding the dotted wedges in the generating functional, see [14,18]. However, the deeper origin of the need for a normal ordering remains hidden. The root of such a reordering will be found in the Hopf algebraic structure of the Grassmann algebra.

## 3. Grassmann–Hopf algebra

We have already introduced the Grassmann algebra above. To turn it into a Hopf algebra, we have to add in a compatible way a co-algebra structure and an antipode. Let us denote the unit map which injects the real or complex field into the algebra with $\eta$, $\eta : \Bbbk \mapsto \bigwedge V$, while $\wedge \equiv m_{\wedge}$, $m_{\wedge} : \bigwedge V \otimes \bigwedge V \mapsto \bigwedge V$ is the product map. Let us furthermore denote the linear space underlying the Grassmann algebra as $W$: thus $W = \langle \bigwedge V \rangle$. Then we can describe the Grassmann algebra by the triple $\bigwedge V = (W, m_{\wedge}, \eta)$. The co-algebra structure is then given by a *diagonalization* $\Delta$—also called co-product—and a co-unit $\epsilon$ which arise naturally from 'dualizing' the algebra structure in a functorial sense, see [27, 38]. The compatibility of algebra and co-algebra structure requires the diagonalization and co-unit to be algebra homomorphisms. That is we require that

$$\epsilon(\mathrm{Id}) = 1 \qquad \epsilon(j_I) = 0 \qquad \epsilon(A \wedge B) = \epsilon(A) \wedge \epsilon(B), \tag{22}$$

while the co-product has to fulfil

$$\Delta(\mathrm{Id}) = \mathrm{Id} \otimes \mathrm{Id} \qquad \Delta(j_I) = j_I \otimes \mathrm{Id} + \mathrm{Id} \otimes j_I \qquad \Delta(A \wedge B) = \Delta(A) \wedge \Delta(B) \tag{23}$$

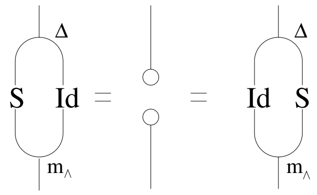


**Figure 1.** Tangle definition of the antipode axioms.

obeying an $\mathbb{Z}_2$-graded tensor product. We also introduce the Sweedler notation of co-products as [38]

$$\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)} \tag{24}$$

where we omit the subscript at the sum sign or even the sum sign itself, which is then implicit. The above wedge product for *tensors*—$(a \otimes \cdots \otimes b) \wedge (c \otimes \cdots \otimes d)$—needs for its evaluation the graded switch $\tau_\wedge$ which is defined as $(u \in \bigwedge_r V, v \in \bigwedge_s V)$

$$\begin{aligned} &\tau_\wedge : \bigwedge_r V \otimes \bigwedge_s V \mapsto \bigwedge_s V \otimes \bigwedge_r V \\ &\tau_\wedge(u \otimes v) = (-1)^{rs}(v \otimes u) \end{aligned} \tag{25}$$

extended by linearity to all elements of $\otimes^2 W$. Finally, we define the antipode $S^\wedge \equiv S \in \mathrm{End}(W)$—i.e. the convolutive inverse of Id—as

$$S^\wedge : \bigwedge_r V \mapsto \bigwedge_r V \qquad S^\wedge(x) = (-1)^r x \tag{26}$$

also extended by linearity. Note that this is exactly the grade involution of the Grassmann algebra. The sextuple

$$H_\wedge := (W, m_\wedge, \eta, \Delta, \epsilon, S^\wedge) \tag{27}$$

is the Grassmann–Hopf algebra, which is in a certain sense unique (universality property), see [38]. It can be checked that, with the above definitions, all axioms for a Hopf algebra are met. In particular, the antipode axioms are fulfilled (see figure 1):

$$\begin{aligned} &\text{(i)} \quad \wedge \circ (S \otimes \mathrm{Id}) \circ \Delta = \eta \circ \epsilon \\ &\text{(ii)} \quad \wedge \circ (\mathrm{Id} \otimes S) \circ \Delta = \eta \circ \epsilon. \end{aligned} \tag{28}$$

Observe that in our case $S^2 = \mathrm{Id}$ and the Grassmann–Hopf algebra is $\mathbb{Z}_2$-graded co-commutative.

## 4. The Rota–Stein Cliffordization formula as Wick theorem

In [34] Rota and Stein introduced a deformed product on Hopf algebras. Adding the structure of a Laplace pairing—for definitions see below—one can add to the universal structure of a Grassmann–Hopf algebra a new product which relies on the Laplacian pairing. This corresponds to an algebra deformation of the Grassmann algebra into a Clifford algebra. We will, however, use this mechanism to connect Grassmann–Hopf algebras which are *different*, i.e. non-isomorphic, as $\mathbb{Z}$-graded (Hopf) algebras. This can be done using a Laplace pairing w.r.t. an antisymmetric form, which will be afterwards defined to be the propagator of the theory.

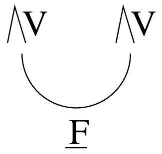


**Figure 2.** Tangle for the scalar-valued pairing.

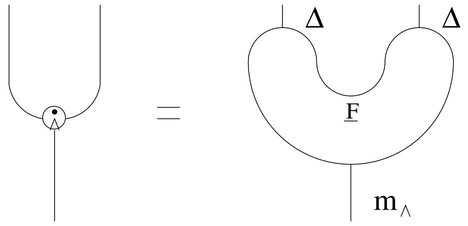


**Figure 3.** Tangle definition of the 'Rota sausage'.

We define a Laplace pairing following [34]. While Rota and Stein have a general pairing $(\cdot|\cdot) : H_\wedge \times H_\wedge \mapsto H_\wedge$, we restrict the target to $\Bbbk$. Let $(w|w')$ be a bilinear mapping from elements $w$, $w'$ of the Grassmann–Hopf algebra $H_\wedge$ into $\Bbbk$ which satisfies the Laplace identities

$$\begin{aligned}
&\text{(i)} \quad (w \wedge w'|w'') = \sum \pm(w|w''_{(1)}) \wedge (w|w''_{(2)}) \\
&\text{(ii)} \quad (w|w' \wedge w'') = \sum \pm(w_{(1)}|w') \wedge (w_{(2)}|w'') \\
&\text{(iii)} \quad (w|w')_{(1)} \wedge ((w|w')_{(2)}|w'') = \sum \pm(w_{(1)}|w'_{(1)}) \wedge ((w_{(2)}|w'_{(2)})|w'') \\
&\text{(iv)} \quad (w|(w'|w'')_{(1)}) \wedge (w'|w'')_{(2)} = \sum \pm(w|(w'_{(1)}|w''_{(1)})) \wedge (w'_{(2)}|w''_{(2)}).
\end{aligned} \tag{29}$$

The signs $\pm$ have to be chosen due to the action of the graded switch to produce the correct permutations. If the bilinear form is scalar-valued, i.e. in $\Bbbk$ as we assume, the wedge products can be safely removed. Relations (i) and (ii) are the Hopf algebraic expression of the Laplace expansion of determinants. Relations (iii) and (iv) state the compatibility of the bilinear form and the co-product being an algebra homomorphism.

We use tangles [40] to make some of the relations more feasible. The tangle for the scalar-valued pairing is illustrated in figure 2.

We are now ready to define with Rota and Stein the deformed product as

$$w \dot{\wedge} w' := \sum \pm w_{(1)} \wedge (w_{(2)}|w'_{(1)})_F \wedge w'_{(2)} \tag{30}$$

or in terms of tangles (see figure 3) as the so-called 'Rota sausage' [31]. Note that the deformed product is given by a *non-local* graph. This process is called *Cliffordization* since it is usually used to introduce a non-trivial symmetric bilinear form. The non-locality of the 'sausage' might have implications on Feynman diagrams built up from such products. Indeed, we found in [19] that some singularities which arise due to reordering in vertices of nonlinear spinor field models disappear due to this rearrangement. Hopf algebras have been recently used successfully in perturbative renormalization theory [8, 24, 33].

Note the perplexing fact that this definition of the dotted wedge product is generic for *any* element of the underlying space $W = \langle \bigwedge V \rangle$ of Grassmann polynomials. This contrasts strongly with the *recursive* definition of the dotted wedge product due to Chevalley deformation of the Grassmann wedge product [6, 7]. There one has ($x \in V$)

$$\gamma_x : \bigwedge V \mapsto \bigwedge V \qquad x \mapsto \gamma_x := x \underset{F}{\lrcorner} + x \wedge \tag{31}$$

which works *only* for $x \in V$ and has to be extended by the rules given above for the Schwinger sources (11). At this point we note that the grade involution appearing in (11)(ii) is exactly the antipode $S^{\wedge}$.

The only weak point in our definition is that we have not yet given a computational definition of the pairing. The pairing can be axiomatized [10]. However, the most important relations are the two Laplace expansions (29)(i) and (29)(ii) which allow us by applying the co-product to decompose the pairing into pairings which contain only elements of the space $V$. The Laplace pairing is a pairing which allows exactly such a decomposition.

However, we will give a second definition of the pairing, which is equivalent to the above one, but might be more familiar to physicists. We want nevertheless to stress that the evaluation of this expression is done by applying the Laplace expansion rules given above, despite the fact that this time the Hopf algebraic nature of the expansion is disguised.

Let $w_r = x_1 \wedge \cdots \wedge x_r$ and $w'_s = y'_s \wedge \cdots \wedge y'_1$ with $x_i, y_j \in V$, define

$$(w_r|w'_s)_F := \begin{cases} \det(x_i|y'_j)_F & r = s \\ 0 & \text{otherwise} \end{cases} \tag{32}$$

and extend it by linearity to the whole space $W$. This can be rewritten using the projection onto the scalar part $\langle\cdot\rangle_0^{\wedge} : W \mapsto \Bbbk$ and the contraction $\lrcorner_F$ w.r.t. $F$ as

$$\begin{aligned}(w|w') &= \langle w \underset{F}{\lrcorner} w'\rangle_0^{\wedge} \\ &= \epsilon(w \underset{F}{\lrcorner} w').\end{aligned} \tag{33}$$

Note that the projection onto the scalars $\langle\cdot\rangle_0^{\wedge}$ is exactly the co-unit $\epsilon$ by definition.

The following example shows how this mechanism works. They have been produced for higher dimensions using the computer algebra packages CLIFFORD for Maple, developed by Rafał Abłamowicz [2, 3] and BIGEBRA [4]. However, the example given below can still be done easily by hand.

**Example 2 (Hopf algebraic Wick reordering).** Let $x, y \in V$, let $\circ$ denote the concatenation of operations and calculate the 'sausage' (i.e. formula (30)):

$$\begin{aligned}x \dot{\wedge} y = {} & \wedge \otimes \wedge \circ (\mathrm{Id} \otimes (\cdot|\cdot) \otimes \mathrm{Id}) \circ (\Delta \otimes \Delta)(x \otimes y) \\ = {} & \wedge \otimes \wedge \circ (\mathrm{Id} \otimes (\cdot|\cdot) \otimes \mathrm{Id})(x \otimes \mathrm{Id} \otimes y \otimes \mathrm{Id} \\ & + x \otimes \mathrm{Id} \otimes \mathrm{Id} \otimes y + \mathrm{Id} \otimes x \otimes y \otimes \mathrm{Id} + \mathrm{Id} \otimes x \otimes \mathrm{Id} \otimes y) \\ = {} & \wedge \otimes \wedge\, (0 + (\mathrm{Id}|\mathrm{Id})_F x \otimes y + (x|y)_F \mathrm{Id} \otimes \mathrm{Id} + 0) \\ = {} & x \wedge y + F_{x,y}\mathrm{Id}.\end{aligned} \tag{34}$$

Thus we obtain as the worked out Rota sausage the relation (13), of the dotted wedge, expressed in undotted wedges. Let us compute a product of three dotted wedges ($x_i \in V$):

$$\begin{aligned}x_1 \dot{\wedge} x_2 \dot{\wedge} x_3 &= x_1 \dot{\wedge} (F_{23}\mathrm{Id} + x_2 \wedge x_3) \\ &= x_1 \wedge x_2 \wedge x_3 + x_1 F_{23} + x_2 F_{31} + x_3 F_{12}.\end{aligned} \tag{35}$$

It seems to be that we have to resolve the tangle recursively, but this is for demonstration only. One can write a closed formula using nested Rota sausages to obtain the result directly. In fact, we see clearly how the $\tau_n$ and $\phi_n$ correlation functions are interrelated due to this expansion. As a last calculation, we find the dotted wedge product of two elements of $x_1 \wedge x_2$, and $x_3 \wedge x_4$, both $\in \wedge^2 V$:

$$\begin{aligned}&(x_1 \wedge x_2) \dot{\wedge} (x_3 \wedge x_4) \\ &\quad = (\wedge \otimes \wedge) \circ (\mathrm{Id} \otimes (\cdot|\cdot) \otimes \mathrm{Id}) \circ (\Delta \otimes \Delta)((x_1 \wedge x_2) \otimes (x_3 \wedge x_4))\end{aligned}$$

$$
\begin{aligned}
&= (\wedge \otimes \wedge) \circ (\mathrm{Id} \otimes (\cdot|\cdot) \otimes \mathrm{Id})((x_1 \wedge x_2) \otimes \mathrm{Id} \otimes (x_3 \wedge x_4) \otimes \mathrm{Id} \\
&\quad + (x_1 \wedge x_2) \otimes \mathrm{Id} \otimes x_3 \otimes x_4 - (x_1 \wedge x_2) \otimes \mathrm{Id} \otimes x_4 \otimes x_3 \\
&\quad + (x_1 \wedge x_2) \otimes \mathrm{Id} \otimes \mathrm{Id} \otimes (x_3 \wedge x_4) + x_1 \otimes x_2 \otimes (x_3 \wedge x_4) \otimes \mathrm{Id} \\
&\quad + x_1 \otimes x_2 \otimes x_3 \otimes x_4 - x_1 \otimes x_2 \otimes x_4 \otimes x_3 \\
&\quad + x_1 \otimes x_2 \otimes \mathrm{Id} \otimes (x_3 \wedge x_4) - x_2 \otimes x_1 \otimes (x_3 \wedge x_4) \otimes \mathrm{Id} \\
&\quad - x_2 \otimes x_1 \otimes x_3 \otimes x_4 + x_2 \otimes x_1 \otimes x_4 \otimes x_3 \\
&\quad - x_2 \otimes x_1 \otimes \mathrm{Id} \otimes (x_3 \wedge x_4) + \mathrm{Id} \otimes (x_1 \wedge x_2) \otimes (x_3 \wedge x_4) \otimes \mathrm{Id} \\
&\quad + \mathrm{Id} \otimes (x_1 \wedge x_2) \otimes x_3 \otimes x_4 - \mathrm{Id} \otimes (x_1 \wedge x_2) \otimes x_4 \otimes x_3 \\
&\quad + \mathrm{Id} \otimes (x_1 \wedge x_2) \otimes \mathrm{Id} \otimes (x_3 \wedge x_4)) \\
&= (\mathrm{Id}|\mathrm{Id})_F x_1 \wedge x_2 \wedge x_3 \wedge x_4 + (x_2|x_3)_F x_1 \wedge x_2 - (x_2|x_4)_F x_1 \wedge x_3 \\
&\quad - (x_1|x_3)_F x_2 \wedge x_4 + (x_1|x_4)_F x_2 \wedge x_3 (x_1 \wedge x_2|x_3 \wedge x_4)_F \mathrm{Id}.
\end{aligned} \tag{36}
$$

It is obvious that one can define an inverse mapping in the same fashion as long as the pairing is non-degenerate. This allows one to expand the undotted wedges into the dotted ones using the Rota–Stein Cliffordization w.r.t. the bilinear form $-F$. This is equivalent to the expansion of the normal-ordered $\phi_n$ correlation functions in terms of the time-ordered $\tau_n$ correlation functions.

**Remark.** In spite of the fact that the Grassmann algebras w.r.t. the two wedges (dotted and undotted) are isomorphic as $\mathbb{Z}$-graded Grassmann algebras—forgetting about the co-algebra structure—this is *not* the case for the Hopf algebras $H_\wedge$ and $H_{\dot\wedge}$. This can be seen easily by examining the co-unit and co-product. Evaluating the co-unit $\epsilon_\wedge$ on a dotted and undotted wedge (for example, from equation (13)) $(x, y \in V)$:

$$
\begin{aligned}
&\epsilon_\wedge(\mathrm{Id}) = 1 = \epsilon_{\dot\wedge}(\mathrm{Id}) \qquad \epsilon_\wedge(x) = 0 = \epsilon_{\dot\wedge}(x) \\
&\epsilon_\wedge(x \wedge y) = 0 \qquad \epsilon_\wedge(x \dot\wedge y) = \epsilon_\wedge(x \wedge y + F_{x,y}\mathrm{Id}) = F_{x,y} \neq 0.
\end{aligned} \tag{37}
$$

Obviously, we find that $\epsilon_\wedge$ is *not* the co-unit of $H_{\dot\wedge}$ and that we are forced to introduce a new co-unit $\epsilon_{\dot\wedge}$ there. An analogous situation is found for the co-products, which are also not identical. We conclude that the Wick isomorphism is an algebra but not a co-algebra isomorphism, since it is not even a co-algebra homomorphism. This opens the interesting possibility of having a family of co-units and co-products for a—up to isomorphy—given Grassmann algebra to make it into a Grassmann–Hopf algebra. In [16] we showed how such a process can be used to introduce different vacua and even condensation phenomena. We know from our present paper that the term 'vacuum' is connected with the co-unit of a Hopf algebra. In fact, the co-unit is a sort of expectation value of the algebra elements which constitute the operators. The Hopf algebraic non-isomorphic $\mathbb{Z}$ gradings are responsible for this feature.

We conjecture, furthermore, that this process is involved in the recent development of Connes and Kreimer [8, 24, 33] of a theory on perturbative renomalization of quantum field theory, where the antipode generates all the counter-terms in the forest theorems. However, their theory uses the $\mathbb{Z}_2$ grading only and is thus not sensitive to the finer $\mathbb{Z}$ grading used in our paper.

## 5. Conclusion

We have shown that the process of Wick reordering is governed by the Grassmann–Hopf algebra structure uniquely assigned to Schwinger sources and a definite wedge product. Introducing a dotted wedge product, using Rota–Stein Cliffordization w.r.t. an antisymmetric bilinear form $F$ and the therefrom induced pairing, we found a closed formula for reordering time- into normal-ordered and normal- into time-ordered $n$-point correlation functions. The Hopf algebraic nature of this process was exhibited and its importance in quantum field theory was demonstrated. It was also demonstrated that the $\mathbb{Z}$ grading of a Grassmann–Hopf algebra is *not* preserved under such a transition. This was connected to different vacua underlying the particular theory. The Stumpf approach to non-perturbative normal ordering was given and compared to the Hopf algebraic method.

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